APPENDIX

I. Detailed Derivations for MINQUE and MIVQUE Estimation of Variance Components in the Unbalanced 1-way Classification Model

1. MIVQUE for the model with $\mu \equiv 0$

The model may be written as

$$y = \left(\sum_i{}^{+} 1_{n_i}\right)\underline{a} + I_N \underline{e} = Z_1 \underline{a} + Z_2 \underline{e} \quad .$$

$$\text{Then } V_1 = Z_1 Z_1' = \sum_i{}^{+} J_{n_i} \quad ,$$

$$V_2 = Z_2 Z_2' = I_N \quad ,$$

$$\text{and } \quad V = \sigma_a^2 V_1 + \sigma_e^2 V_2 = \sum_i{}^{+} \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right) \quad . \tag{A1}$$

Under the assumption that $\mu \equiv 0$, one operates with Rao's general expressions for MINQUE and MIVQUE taking $X = 0$ and therefore $P_1 = X(X'V^{-1}X)^{-1}X'V^{-1} = 0$.

Then
$$R_1 = V^{-1}(I_V - P_1) = V^{-1}$$

$$= \sum{}^{+}\left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{\sigma_a^2}{\sigma_e^2(\sigma_e^2 + n_i\sigma_a^2)} J_{n_i}\right] \quad . \tag{A2}$$

We need to derive $s_{1:11} = tr(V_1 R_1)^2$, $s_{1:12} = tr(V_2 R_1 V_1 R_1)$, $s_{1:22} = tr(V_2 R_1)^2$, $u_{1:1} = y'R_1 V_1 R_1 y$, and $u_{1:2} = y'R_1 V_2 R_1 y$.

From (A1) and (A2)

$$V_1 R_1 = \sum^{+} \left[\frac{1}{\sigma_e^2} J_{n_i} - \frac{n_i \sigma_a^2}{\sigma_e^2(\sigma_e^2 + n_i \sigma_a^2)} J_{n_i} \right]$$

$$= \sum^{+} \frac{1}{\sigma_e^2 + n_i \sigma_a^2} J_{n_i} \quad ;$$

$$R_1 V_1 R_1 = \sum^{+} \left[\frac{1}{\sigma_e^2(\sigma_e^2 + n_i \sigma_a^2)} J_{n_i} - \frac{n_i \sigma_a^2}{\sigma_e^2(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_i} \right]$$

$$= \sum^{+} \frac{1}{(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_i} \quad ; \tag{A3}$$

$$(V_1 R_1)^2 = \sum^{+} \frac{n_i}{(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_i} \quad ; \tag{A4}$$

$$V_2 R_1 = R_1 = V^{-1} \text{ as given in (A2)} \quad ;$$

$$R_1 V_2 R_1 = (V_2 R_1)^2 = R_1^2$$

$$= \sum^{+} \left[\frac{1}{\sigma_e^4} I_{n_i} + \left(\frac{-2\sigma_a^2}{\sigma_e^4(\sigma_e^2 + n_i \sigma_a^2)} + \frac{n_i \sigma_a^4}{\sigma_e^4(\sigma_e^2 + n_i \sigma_a^2)^2} \right) J_{n_i} \right]$$

$$= \sum^{+} \left[\frac{1}{\sigma_e^4} I_{n_i} - \frac{2\sigma_e^2 \sigma_a^2 + n_i \sigma_a^4}{\sigma_e^4(\sigma_e^2 + n_i \sigma_a^2)^2} J_{n_i} \right] \quad ; \tag{A5}$$

and $R_1 V_1 R_1 V_2 = R_1 V_1 R_1$ of (A3) . (A6)

In what follows it is convenient to let

$$q_i = \frac{1}{1 + n_i \dfrac{\sigma_a^2}{\sigma_e^2}} = \frac{\sigma_e^2}{\sigma_e^2 + n_i \sigma_a^2} \quad . \tag{A7}$$

This notation duplicates that of Townsend and Searle (1971) and facilitates showing the equivalence of their results and those obtained here.

Using (A4), (A5), and (A6) gives

$$s_{1:11} = \mathrm{tr}(V_1 R_1)^2 = \sum \frac{n_i^2}{(\sigma_e^2 + n_i\sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum n_i^2 q_i^2 \quad , \tag{A8}$$

$$s_{1:12} = \mathrm{tr}(V_1 R_1 V_2 R_1) = \sum \frac{n_i}{(\sigma_e^2 + n_i\sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum n_i q_i^2 \quad , \tag{A9}$$

and $s_{1:22} = \mathrm{tr}(V_2 R_1)^2 = \sum \left[\dfrac{n_i\sigma_e^4 + 2n_i(n_i - 1)\sigma_e^2\sigma_a^2 + n_i^2(n_i - 1)\sigma_a^4}{\sigma_e^4(\sigma_e^2 + n_i\sigma_a^2)^2} \right]$

$$= \frac{1}{\sigma_e^4} \sum q_i^2 \left[n_i + 2n_i(n_i - 1)\frac{\sigma_a^2}{\sigma_e^2} + n_i^2(n_i - 1)\frac{\sigma_a^4}{\sigma_e^4} \right]$$

$$= \frac{1}{\sigma_e^4} \sum q_i^2 \left[(n_i - 1)\left(1 + n_i\frac{\sigma_a^2}{\sigma_e^2}\right)^2 + 1 \right]$$

$$= \frac{1}{\sigma_e^4} \sum q_i^2 \left[(n_i - 1)q_i^{-2} + 1 \right]$$

$$= \frac{1}{\sigma_e^4} \left[\sum q_i^2 + N - a \right] \quad . \tag{A10}$$

From (A3) and (A5)

$$u_{1:1} = y' R_1 V_1 R_1 y = \sum_i \frac{y_{i.}^2}{(\sigma_e^2 + n_i\sigma_a^2)^2} = \frac{1}{\sigma_e^4} \sum_i q_i^2 y_{i.}^2 . \tag{A11}$$

and $u_{1:2} = y' R_1 V_2 R_1 y$

$$= \sum_i \sum_j \frac{y_{ij}^2}{\sigma_e^4} - \sum_i \frac{2\sigma_e^2\sigma_a^2 + n_i\sigma_a^4}{\sigma_e^4(\sigma_e^2 + n_i\sigma_a^2)^2} y_{i.}^2$$

$$= \frac{1}{\sigma_e^4}\left[\sum_i \sum_j y_{ij}^2 - \sum_i \left\{2\frac{\sigma_a^2}{\sigma_e^2} q_i^2 + n_i \frac{\sigma_a^4}{\sigma_e^4} q_i^2\right\} y_{i.}^2\right]$$

$$= \frac{1}{\sigma_e^4}\left[\sum\sum y_{ij}^2 - \sum q_i^2 \left\{\frac{1}{n_i}\left(1 + n_i \frac{\sigma_a^2}{\sigma_e^2}\right)^2 - \frac{1}{n_i}\right\} y_{i.}^2\right]$$

$$= \frac{1}{\sigma_e^4}\left[\sum\sum y_{ij}^2 - \sum q_i^2 \left\{\frac{1}{n_i} q_i^{-2} - \frac{1}{n_i}\right\} y_{i.}^2\right]$$

$$= \frac{1}{\sigma_e^4}\left[\sum_i \sum_j y_{ij}^2 - \sum_i \frac{y_{i.}^2}{n_i} + \sum_i \frac{q_i^2 y_{i.}^2}{n_i}\right] . \tag{A12}$$

Then the estimators of σ_e^2 and σ_a^2 are

$$\hat{\sigma}_{1e}^2 = \frac{1}{\Delta_1}\left[- s_{1:12}u_{1:1} + s_{1:11}u_{1:2} \right] \tag{A13}$$

and

$$\hat{\sigma}_{1a}^2 = \frac{1}{\Delta_1}\left[s_{1:22}u_{1:1} - s_{1:12}u_{1:2} \right] \tag{A14}$$

using (A8) through (A12) and letting $\Delta_1 = s_{1:11}s_{1:22} - s_{1:12}^2$ = the determinant of S_1 . Equations (A13) and (A14) may be written in terms of (A8) through (A12) as

$$\hat{\sigma}_{1e}^2 = \frac{-\left[\frac{1}{\sigma_e^4}\sum n_i q_i^2\right]\left[\frac{1}{\sigma_e^4}\sum q_i^2 y_{i.}^2\right] + \left[\frac{1}{\sigma_e^4}\sum n_i^2 q_i^2\right]\left[\frac{1}{\sigma_e^4}\left(\sum\sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n_i} + \sum \frac{q_i^2 y_{i.}^2}{n_i}\right)\right]}{\left[\frac{1}{\sigma_e^4}\sum n_i^2 q_i^2\right]\left[\frac{1}{\sigma_e^4}\left(\sum q_i^2 + N - a\right)\right] - \left[\frac{1}{\sigma_e^4}\sum n_i q_i^2\right]^2}$$

$$= \frac{\sum\left[\sum n_i^2 q_i^2 - n_i \sum n_i q_i^2\right] \frac{q_i^2 y_{i\cdot}^2}{n_i} + \sum n_i^2 q_i^2 \left[\sum\sum y_{ij}^2 - \sum \frac{y_{i\cdot}^2}{n_i}\right]}{\left[\sum n_i^2 q_i^2\right]\left[\sum q_i^2 + N - a\right] - \left[\sum n_i q_i^2\right]^2} \tag{A15}$$

$$\text{and } \hat{\sigma}_{1a}^2 = \frac{\frac{1}{\sigma_e^4}\left[\sum q_i^2 + N - a\right]\left[\frac{1}{\sigma_e^4}\sum q_i^2 y_{i\cdot}^2\right] - \left[\frac{1}{\sigma_e^4}\sum n_i q_i^2\right]\left[\frac{1}{\sigma_e^4}\left(\sum\sum y_{ij}^2 - \sum \frac{y_{i\cdot}^2}{n_i} + \sum \frac{q_i^2 y_{i\cdot}^2}{n_i}\right)\right]}{\left[\frac{1}{\sigma_e^4}\sum n_i^2 q_i^2\right]\left[\frac{1}{\sigma_e^4}\left(\sum q_i^2 + N - a\right)\right] - \left[\frac{1}{\sigma_e^4}\sum n_i q_i^2\right]^2}$$

$$= \frac{\sum\left[n_i\left(\sum q_i^2 + N - a\right) - \sum n_i q_i^2\right] \frac{q_i^2 y_{i\cdot}^2}{n_i} - \sum n_i q_i^2 \left[\sum\sum y_{ij}^2 - \sum \frac{y_{i\cdot}^2}{n_i}\right]}{\left[\sum n_i^2 q_i^2\right]\left[\sum q_i^2 + N - a\right] - \left[\sum n_i q_i^2\right]^2} \ . \tag{A16}$$

Equations (A15) and (A16) are identical respectively to equations (2) and (4) of Townsend and Searle (1971) for these estimators.

Using the results that $v(y'Ay) = 2\ \mathrm{tr}(VA)^2$ and $\mathrm{cov}(y'Ay,\ y'By) = 2\ \mathrm{tr}(VAVB)$ when $y \sim N(0,\ V)$, the variances and covariance of the estimators can be written as

$$v\left(\hat{\sigma}_{1e}^2\right) = \frac{1}{\Delta_1^2}\left[s_{1:12}^2 v(u_{1:1}) + s_{1:11}^2 v(u_{1:2}) - 2s_{1:11}s_{1:12}\mathrm{cov}(u_{1:1},u_{1:2})\right] , \tag{A17}$$

$$v\left(\hat{\sigma}_{1a}^2\right) = \frac{1}{\Delta_1^2}\left[s_{1:22}^2 v(u_{1:1}) + s_{1:12}^2 v(u_{1:2}) - 2s_{1:12}s_{1:22}\mathrm{cov}(u_{1:1},u_{1:2})\right] , \tag{A18}$$

$$\text{and} \quad \mathrm{cov}\left(\hat{\sigma}_{1e}^2,\hat{\sigma}_{1a}^2\right) = \frac{1}{\Delta_1^2}\left[-s_{1:12}s_{1:22}v(u_{1:1}) - s_{1:11}s_{1:12}v(u_{1:2}) + (s_{1:11}s_{1:22} + s_{1:12}^2)\mathrm{cov}(u_{1:1},u_{1:2})\right] , \tag{A19}$$

leaving only $v(u_{1:1})$, $v(u_{1:2})$, and $\mathrm{cov}(u_{1:1},u_{1:2})$ to be derived. Noting that $R_1 = V^{-1}$,

$$v\left(u_{1:1}\right) = 2tr(VR_1V_1R_1)^2 = 2tr(V_1R_1)^2 = 2s_{1:11} , \tag{A20}$$

$$v\left(u_{1:2}\right) = 2tr(VR_1V_2R_1)^2 = 2tr(V_2R_1)^2 = 2s_{1:22} , \tag{A21}$$

and $$\operatorname{cov}\left(u_{1:1},u_{1:2}\right) = 2tr(VR_1V_1R_1VR_1V_2R_1) = 2tr(V_1R_1V_2R_1) = 2s_{1:12} , \tag{A22}$$

these results coming from (A8), (A9), and (A10).

Substituting (A20), (A21), and (A22) into (A17), (A18), and (A19) yields

$$\begin{aligned} v\left(\hat{\sigma}^2_{1e}\right) &= \frac{2}{\Delta_1^2}\left[s^2_{1:12}s_{1:11} + s^2_{1:11}s_{1:22} - 2s_{1:11}s^2_{1:12}\right] \\ &= \frac{2s_{1:11}\Delta_1}{\Delta_1^2} = \frac{2s_{1:11}}{\Delta_1} \end{aligned} \tag{A23}$$

and similarly

$$v\left(\hat{\sigma}^2_{1a}\right) = \frac{2s_{1:22}}{\Delta_1} \tag{A24}$$

and $$\operatorname{cov}\left(\hat{\sigma}^2_{1e},\hat{\sigma}^2_{1a}\right) = \frac{-2s_{1:12}}{\Delta_1} . \tag{A25}$$

Substituting (A8), (A9), and (A10) into (A23) and (A24) immediately displays the forms for $v\left(\hat{\sigma}^2_{1e}\right)$ and $v\left(\hat{\sigma}^2_{1a}\right)$ given in equations (3) and (5) of Townsend and Searle (1971).

2. MINQUE for the model with $\mu \equiv 0$

The minimization for this case differs from MIVQUE only in the substitution of

$$V_u = V_1 + V_2 \tag{A26}$$

for $V = \sigma_a^2 V_1 + \sigma_e^2 V_3$ in (A2). Therefore, although the estimators could be derived by a series of steps completely analogous to those used for MIVQUE, it is simpler to obtain the estimators as a special case of MIVQUE, setting $\sigma_e^2 = \sigma_a^2 = 1$ in equations (A2) through (A14). Doing this in (A7) we then rewrite q_i as w_i where

$$w_i = \frac{1}{1 + n_i} \quad . \tag{A27}$$

From (A8) through (A12) we get

$$s_{2:11} = \sum \frac{n_i^2}{(1 + n_i)^2} = \sum n_i^2 w_i^2 \ , \tag{A28}$$

$$s_{2:12} = \sum \frac{n_i}{(1 + n_i)^2} = \sum n_i w_i^2 \ , \tag{A29}$$

$$s_{2:22} = \sum \frac{1}{(1 + n_i)^2} + N - a$$

$$= \sum w_i^2 + N - a \ , \tag{A30}$$

$$u_{2:1} = \sum \frac{y_i^2}{(1 + n_i)^2} = \sum w_i^2 y_{i.}^2 \ , \tag{A31}$$

and

$$u_{2:2} = \sum \sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n_i} + \sum \frac{y_{i.}^2}{n_i(1 + n_i)^2}$$

$$= \sum \sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n_i} + \sum \frac{w_i^2 y_{i.}^2}{n_i} \quad . \tag{A32}$$

Letting $\Delta_2 = s_{2:11}s_{2:22} - s_{2:12}^2$ = the determinant of S_2 and using (A28) through (A32), the estimators may be written as

$$\hat{\sigma}_{2e}^2 = \frac{1}{\Delta_2}\left[-s_{2:12}u_{2:1} + s_{2:11}u_{2:2}\right] \tag{A33}$$

and $$\hat{\sigma}^2_{2a} = \frac{1}{\Delta_2}\left[s_{2:22}u_{2:1} - s_{2:12}\,u_{2:2}\right] . \tag{A34}$$

The variances and covariance of the estimators must be derived directly. They take the form

$$v\left(\hat{\sigma}^2_{2e}\right) = \frac{1}{\Delta_2^2}\left[s^2_{2:12}v(u_{2:1}) + s^2_{2:11}v(u_{2:2}) - 2s_{2:11}s_{2:12}\mathrm{cov}(u_{2:1},u_{2:2})\right] , \tag{A35}$$

$$v\left(\hat{\sigma}^2_{2a}\right) = \frac{1}{\Delta_2^2}\left[s^2_{2:22}v(u_{2:1}) + s^2_{2:12}v(u_{2:2}) - 2s_{2:12}s_{2:22}\mathrm{cov}(u_{2:1},u_{2:2})\right] , \tag{A36}$$

and $$\mathrm{cov}\left(\hat{\sigma}^2_{2e},\hat{\sigma}^2_{2a}\right) = \frac{1}{\Delta_2^2}\left[-s_{2:12}s_{2:22}v(u_{2:1}) - s_{2:11}s_{2:12}v(u_{2:2}) + (s_{2:11}s_{2:22} + s^2_{2:12})\mathrm{cov}(u_{2:1},u_{2:2})\right] , \tag{A37}$$

where $v(u_{2:1})$, $v(u_{2:2})$, and $\mathrm{cov}(u_{2:1},u_{2:2})$ are given below in (A39), (A41), and (A42).

In the following we will use

$$R_2V_1R_2 = \sum{}^{+}\, w_i^2 J_{n_i}$$

and $$R_2V_2R_2 = \sum{}^{+}\left[I_{n_i} - w_i^2(n_i + 2)J_{n_i}\right] ,$$

obtained from (A3) and (A5) by setting $\sigma_e^2 = \sigma_a^2 = 1$, and q_i as defined in (A7).

$$v(u_{2:1}) = 2\mathrm{tr}(VR_2V_1R_2)^2 \quad \text{where}$$

$$VR_2V_1R_2 = \sum{}^{+}\left[w_i^2\sigma_e^2 + w_i^2 n_i \sigma_a^2\right]J_{n_i}$$

$$= \sum{}^{+}\,\frac{w_i^2\sigma_e^2}{q_i}\,J_{n_i} \tag{A38}$$

and $$(VR_2V_1R_2)^2 = \sum^{+} \frac{n_i w_i^4 \sigma_e^4}{q_i^2} J_{n_i} \quad .$$

Then $$v(u_{2:1}) = 2\sigma_e^4 \sum \frac{n_i^2 w_i^4}{q_i^2} \quad . \tag{A39}$$

Similarly,

$$v(u_{2:2}) = 2tr(VR_2V_2R_2)^2 \qquad \text{for}$$

$$VR_2V_2R_2 = \sum^{+}\left[\sigma_e^2 I_{n_i} + \left(\sigma_a^2 - w_i^2\left[n_i + 2\right]\sigma_e^2 - n_i w_i^2\left[n_i + 2\right]\sigma_a^2\right) J_{n_i}\right]$$

$$= \sum^{+}\left[\sigma_e^2 I_{n_i} + w_i^2\left(\sigma_a^2 - \left[n_i + 2\right]\sigma_e^2\right) J_{n_i}\right] \tag{A40}$$

and $$(VR_2V_2R_2)^2 = \sum^{+}\left[\sigma_e^4 I_{n_i} + \left\{2w_i^2\sigma_e^2\left(\sigma_a^2 - \left[n_i + 2\right]\sigma_e^2\right)\right.\right.$$

$$\left.\left. + n_i w_i^4\left(\sigma_a^2 - \left[n_i + 2\right]\sigma_e^2\right)^2\right\} J_{n_i}\right]$$

$$= \sum^{+}\left[\sigma_e^4 I_{n_i} + w_i^4\left(-n_i^3\sigma_e^4 - 4n_i^2\sigma_e^4 - 6n_i\sigma_e^4\right.\right.$$

$$\left.\left. + 2\sigma_e^2\sigma_a^2 - 4\sigma_e^4 + n_i\sigma_a^4\right) J_{n_i}\right]$$

$$= \sum^{+}\left[\sigma_e^4 I_{n_i} + w_i^4\left(\frac{-w_i^{-4}\sigma_e^4 + (\sigma_e^2 + n_i\sigma_a^2)^2}{n_i}\right) J_{n_i}\right]$$

$$= \sum^{+}\left[\sigma_e^4 I_{n_i} - \left(\frac{\sigma_e^4}{n_i} - \frac{w_i^4\sigma_e^4}{n_i q_i^2}\right) J_{n_i}\right] \quad .$$

So $$v(u_{2:2}) = 2\left[N\sigma_e^4 - a\sigma_e^4 + \sum \frac{w_i^4\sigma_e^4}{q_i^2}\right]$$

$$= 2\sigma_e^4\left[\sum \frac{w_i^4}{q_i^2} + N - a\right] \quad . \tag{A41}$$

$$\operatorname{cov}(u_{2:1}, u_{2:2}) = 2\operatorname{tr}(VR_2V_1R_2VR_2V_2R_2) \text{ where from (A38)}$$

and (A40)

$$VR_2V_1R_2VR_2V_2R_2 = \sum{}^{+}\left[\frac{w_i^2\sigma_e^4}{q_i} + \frac{n_i w_i^4 \sigma_e^2}{q_i}\left(\sigma_a^2 - [n_i + 2]\sigma_e^2\right)\right] J_{n_i}$$

$$= \sum{}^{+} \frac{w_i^4\sigma_e^2(\sigma_e^2 + n_i\sigma_a^2)}{q_i} J_{n_i}$$

$$= \sum{}^{+} \frac{w_i^4\sigma_e^4}{q_i^2} J_{n_i} \quad .$$

Thus
$$\operatorname{cov}(u_{2:1}, u_{2:2}) = 2\sigma_e^4 \sum \frac{n_i w_i^4}{q_i^2} \quad . \tag{A42}$$

3. <u>MIVQUE for the model with $\mu \neq 0$</u>

Writing the model as

$$y = \mu 1_N + \left(\sum{}^{+} 1_{n_i}\right) \underset{\sim}{a} + I_N \underset{\sim}{e} = X\mu + Z_1 \underset{\sim}{a} + Z_2 \underset{\sim}{e}$$

implies

$$V_1 = Z_1 Z_1' = \sum{}^{+} J_{n_i} \;,$$

$$V_2 = Z_2 Z_2' = I_N \;,$$

and
$$V = \sigma_a^2 V_1 + \sigma_e^2 V_2 = \sum{}^{+} \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right) \quad .$$

Then
$$V^{-1} = \sum{}^{+}\left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{\sigma_a^2}{\sigma_e^2(\sigma_e^2 + n_i\sigma_a^2)} J_{n_i}\right]$$

and $$X'V^{-1}X = 1_N' V^{-1} 1_N = \sum \frac{n_i}{\sigma_e^2 + n_i\sigma_a^2} \quad .$$

We write

$$k_i = \frac{n_i}{\sigma_e^2 + n_i\sigma_a^2} \text{ and } k = \frac{1}{\sum k_i} \tag{A43}$$

as notational shorthand, with $n_i q_i = k_i \sigma_e^2$,

$$X'V^{-1}X = \frac{1}{k}$$

and $$(X'V^{-1}X)^{-1} = k \quad .$$

Then $$P_3 = X(X'V^{-1}X)^{-1}X'V^{-1}$$

$$= k1_N \left(\frac{1}{\sigma_e^2 + n_1\sigma_a^2} 1'_{n_1} \cdots \frac{1}{\sigma_e^2 + n_a\sigma_a^2} 1'_{n_a} \right)$$

$$= \left[P_{3:1} P_{3:2} \cdots P_{3:a} \right] \text{ where } P_{3:i} = \frac{kk_i}{n_i} J_{N\times n_i} .$$

$R_3 = V^{-1}(I - P_3)$ and its products with V_1 and V_2 have nonzero off-diagonal blocks, unlike the situation where $\mu \equiv 0$. It is convenient to partition these matrices as $M = \{M_{ii'}\}$ where $i,i' = 1, 2, \cdots, a$, and to derive results by developing expressions for typical diagonal and off-diagonal blocks or sub-matrices.

$R_3 = V^{-1}(I_N - P_3)$ has as its i^{th} diagonal block

$$[i^{th} \text{ diagonal block of } V^{-1}][i^{th} \text{ diagonal block of } (I_N - P_3)]$$

$$= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2}{n_i\sigma_e^2} J_{n_i} \right] \left[I_{n_i} - \frac{kk_i}{n_i} J_{n_i} \right]$$

$$= \frac{1}{\sigma_e^2} I_{n_i} - \left[\frac{kk_i}{n_i\sigma_e^2} + \frac{k_i\sigma_a^2}{n_i\sigma_e^2} - \frac{kk_i^2\sigma_a^2}{n_i\sigma_e^2}\right] J_{n_i}$$

$$= \frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2 + kk_i(1 - k_i\sigma_a^2)}{n_i\sigma_e^2} J_{n_i} \tag{A44}$$

and as its i,i'^{th} off-diagonal block

[i^{th} diagonal block of V^{-1}][i,i'^{th} off-diagonal block of $(I_N - P_3)$]

$$= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2}{n_i\sigma_e^2} J_{n_i}\right]\left[- \frac{kk_{i'}}{n_{i'}} J_{n_i \times n_{i'}}\right]$$

$$= \left[- \frac{kk_{i'}}{n_{i'}\sigma_e^2} + \frac{kk_ik_{i'}\sigma_a^2}{n_{i'}\sigma_e^2}\right] J_{n_i \times n_{i'}}$$

$$= - \frac{kk_{i'}\left[1 - k_i\sigma_a^2\right]}{n_{i'}\sigma_e^2} J_{n_i \times n_{i'}}$$

$$= - \frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}} \quad , \tag{A45}$$

making use in the final step of the identity

$$\left(1 - k_i\sigma_a^2\right) = \frac{k_i\sigma_e^2}{n_i} = q_i \tag{A46}$$

which will be used frequently in what follows.

From $V_1 = \Sigma^+ J_{n_i}$ and (A44) $V_1 R_3$ has its i^{th} diagonal block

$$J_{n_i}\left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2 + kk_i\left(1 - k_i\sigma_a^2\right)}{n_i\sigma_e^2} J_{n_i}\right]$$

$$= \frac{(1 - kk_i)(1 - k_i\sigma_a^2)}{\sigma_e^2} J_{n_i} = \frac{k_i(1 - kk_i)}{n_i} J_{n_i} \tag{A47}$$

and from (A45) its i,i'^{th} off-diagonal block

$$J_{n_i}\left[- \frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}}\right] = - \frac{kk_ik_{i'}}{n_{i'}} J_{n_i \times n_{i'}} \quad ; \tag{A48}$$

from (A44) — (A48) $R_3V_1R_3$ has its i^{th} diagonal block

$$[i^{th} \text{ diagonal block of } R_3][i^{th} \text{ diagonal block of } V_1R_3]$$

$$+ \sum_{i' \neq i}^{-} \left[i,i'^{th} \text{ off-diag. block of } R_3\right]\left[i',i^{th} \text{ off-diag. block of } V_1R_3\right]$$

$$= \left[\frac{1}{\sigma_e^2}I_{n_i} - \frac{k_i\sigma_a^2 + kk_i(1 - k_i\sigma_a^2)}{n_i\sigma_e^2} J_{n_i}\right]\left[\frac{k_i(1 - kk_i)}{n_i} J_{n_i}\right]$$

$$+ \sum_{i' \neq i} \left[- \frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}}\right]\left[- \frac{kk_ik_{i'}}{n_i} J_{n_{i'} \times n_i}\right]$$

$$= \frac{k_i(1 - kk_i) - k_i^2\sigma_a^2(1 - kk_i) - kk_i^2(1 - kk_i)(1 - k_i\sigma_a^2)}{n_i\sigma_e^2} J_{n_i}$$

$$+ \sum_{i' \neq i} \frac{k^2k_i^2k_{i'}^2}{n_i^2} J_{n_i}$$

$$= \frac{k_i(1 - kk_i)\dfrac{k_i\sigma_e^2}{n_i} - kk_i^2 \dfrac{k_i\sigma_e^2}{n_i}}{n_i\sigma_e^2} J_{n_i} + \frac{k^2k_i^2\Sigma k_i^2}{n_i^2} J_{n_i}$$

$$= \frac{k_i^2}{n_i^2}\left(1 - 2kk_i + k^2 \sum^{-} k_i^2\right) J_{n_i} \tag{A49}$$

and from (A44) — (A48) its i,i'^{th} off-diagonal block

$[i^{th}$ diagonal block of $R_3][i,i'^{th}$ off-diagonal block of $V_1R_3]$

$+ [i,i'^{th}$ off-diagonal block of $R_3][i'^{th}$ diagonal block of $V_1R_3]$

$+ \sum_{i''\neq i\neq i'} [i,i''^{th}$ off-diag. block of $R_3][i'',i'^{th}$ off-diag. block of $V_1R_3]$

$$= \left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2 + kk_i(1 - k_i\sigma_a^2)}{n_i\sigma_e^2} J_{n_i}\right]\left[-\frac{kk_ik_{i'}}{n_{i'}} J_{n_i \times n_{i'}}\right]$$

$$+ \left[-\frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}}\right]\left[\frac{k_{i'}(1 - kk_{i'})}{n_{i'}} J_{n_{i'}}\right]$$

$$+ \sum_{i''\neq i\neq i'} \left[-\frac{kk_ik_{i''}}{n_in_{i''}} J_{n_i \times n_{i''}}\right]\left[-\frac{kk_{i'}k_{i''}}{n_{i'}} J_{n_{i''} \times n_{i'}}\right]$$

$$= \left[-\frac{kk_ik_{i'}(1 - k_i\sigma_a^2) - k^2k_i^2k_{i'}\dfrac{k_i\sigma_e^2}{n_i}}{n_{i'}\sigma_e^2} - \frac{kk_ik_{i'}^2(1 - kk_{i'})}{n_in_{i'}}\right.$$

$$\left. + \sum_{i''\neq i\neq i'} \frac{k^2k_ik_{i'}k_{i''}^2}{n_in_{i'}}\right] J_{n_i \times n_{i'}}$$

$$= \left[-\frac{kk_ik_{i'}\dfrac{k_i\sigma_e^2}{n_i}}{n_{i'}\sigma_e^2} - \frac{kk_ik_{i'}^2}{n_in_{i'}} + \frac{k^2k_ik_{i'}\Sigma k_i^2}{n_in_{i'}}\right] J_{n_i \times n_{i'}}$$

$$= \frac{k_ik_{i'}\left(k^2\Sigma k_i^2 - k[k_i + k_{i'}]\right)}{n_in_{i'}} J_{n_i \times n_{i'}} \quad ; \qquad \text{(A50)}$$

and from (A47) and (A48) $(V_1R_3)^2$ has its i^{th} diagonal block

$$\left[\frac{k_i(1 - kk_i)}{n_i} J_{n_i}\right]^2 + \sum_{i' \neq i} \left[- \frac{kk_i k_{i'}}{n_{i'}} J_{n_i \times n_{i'}}\right]\left[- \frac{kk_i k_{i'}}{n_i} J_{n_{i'} \times n_i}\right]$$

$$= \left[\frac{k_i^2(1 - kk_i)^2}{n_i} + \sum_{i' \neq i} \frac{k^2 k_i^2 k_{i'}^2}{n_i}\right] J_{n_i}$$

$$= \frac{k_i^2 - 2kk_i^3 + k^2 k_i^2 \Sigma k_i^2}{n_i} J_{n_i}$$

$$= \frac{k_i^2(1 - 2kk_i + k^2 \Sigma k_i^2)}{n_i} J_{n_i} \quad . \tag{A51}$$

Since $V_2 = I$,

$$V_2 R_3 = R_3 \quad . \tag{A52}$$

Then from (A44) and (A45) $R_3 V_2 R_3 = (V_2 R_3)^2 = R_3^2$ has its i^{th} diagonal block

$$\left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i \sigma_a^2 + kk_i(1 - k_i \sigma_a^2)}{n_i \sigma_e^2} J_{n_i}\right]^2 + \sum_{i' \neq i} \left[- \frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_i \times n_{i'}}\right]\left[- \frac{kk_i k_{i'}}{n_i n_{i'}} J_{n_{i'} \times n_i}\right]$$

$$= \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{-2k_i \sigma_a^2 - 2kk_i(1 - k_i \sigma_a^2) + k_i^2 \sigma_a^4 + 2kk_i^2(1 - k_i \sigma_a^2)\sigma_a^2 + k^2 k_i^2 (1 - k_i \sigma_a^2)^2}{n_i \sigma_e^4}\right.$$

$$\left. + \sum_{i' \neq i} \frac{k^2 k_i^2 k_{i'}^2}{n_i^2 n_{i'}}\right] J_{n_i}$$

$$= \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{-2k_i \sigma_a^2 + k_i^2 \sigma_a^4 - 2kk_i(1 - k_i \sigma_a^2) + 2kk_i^2(1 - k_i \sigma_a^2)\sigma_a^2}{n_i \sigma_e^4}\right.$$

$$+ \sum_{\text{all } i'} \frac{k^2 k_i^2 k_{i'}^2}{n_i^2 n_{i'}} \Big] J_{n_i}$$

$$= \frac{1}{\sigma_e^4} I_{n_i} + \left[\frac{(1 - k_i\sigma_a^2)^2 - 1 - 2kk_i(1 \quad k_i\sigma_a^2)^2}{n_i\sigma_e^4} + \frac{k^2k_i^2}{n_i^2} \sum \frac{k_i^2}{n_i} \right] J_{n_i}$$

$$= \frac{1}{\sigma_e^4} I_{n_i} + \left[- \frac{1}{n_i\sigma_e^4} + \frac{k_i^2(1 - 2kk_i)}{n_i^3} + \frac{k^2k_i^2}{n_i^2} \sum \frac{k_i^2}{n_i} \right] J_{n_i} \tag{A53}$$

and also from (A44) and (A45) its i,i'^{th} off-diagonal block

$$\left[\frac{1}{\sigma_e^2} I_{n_i} - \frac{k_i\sigma_a^2 + kk_i(1 - k_i\sigma_a^2)}{n_i\sigma_e^2} J_{n_i}\right]\left[- \frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}}\right]$$

$$+ \left[- \frac{kk_ik_{i'}}{n_in_{i'}} J_{n_i \times n_{i'}}\right]\left[\frac{1}{\sigma_e^2} I_{n_{i'}} - \frac{k_{i'}\sigma_a^2 + kk_{i'}(1 - k_{i'}\sigma_a^2)}{n_{i'}\sigma_e^2} J_{n_{i'}}\right]$$

$$+ \sum_{i'' \neq i \text{ or } i'} \left[- \frac{kk_ik_{i''}}{n_in_{i''}} J_{n_i \times n_{i''}}\right]\left[- \frac{kk_{i'}k_{i''}}{n_{i'}n_{i''}} J_{n_{i''} \times n_{i'}}\right]$$

$$= \Big[\frac{-kk_ik_{i'} + kk_i^2k_{i'}\sigma_a^2 + k^2k_i^2k_{i'}(1 - k_i\sigma_a^2) - kk_ik_{i'} + kk_ik_{i'}^2\sigma_a^2 + k^2k_ik_{i'}^2(1 - k_{i'}\sigma_a^2)}{n_in_{i'}\sigma_e^2}$$

$$+ \sum_{i'' \neq i \text{ or } i'} \frac{k^2k_ik_{i'}k_{i''}^2}{n_in_{i'}n_{i''}} \Big] J_{n_i \times n_{i'}}$$

$$= \left[\frac{-kk_ik_{i'}(1 - k_i\sigma_a^2) - kk_ik_{i'}(1 - k_{i'}\sigma_a^2)}{n_in_{i'}\sigma_e^2} + \sum_{\text{all } i''} \frac{k^2k_ik_{i'}k_{i''}^2}{n_in_{i'}n_{i''}}\right] J_{n_i \times n_{i'}}$$

$$= \left[- \frac{kk_i^2k_{i'}}{n_i^2n_{i'}} - \frac{kk_ik_{i'}^2}{n_in_{i'}^2} + \frac{k^2k_ik_{i'}}{n_in_{i'}} \sum \frac{k_i^2}{n_i}\right] J_{n_i \times n_{i'}} \quad . \tag{A54}$$

The elements of S_3 and U_3 can now be obtained. From (A51)

$$s_{3:11} = tr(V_1R_3)^2 = \sum n_i\left[\frac{k_i^2(1 - 2kk_i + k^2\Sigma k_i^2)}{n_i}\right]$$

$$= \sum k_i^2 - 2k \sum k_i^3 + k^2\left(\sum k_i^2\right)^2 \quad ; \tag{A55}$$

from (A49)

$$s_{3:12} = tr(V_1R_3V_2R_3) = tr(R_3V_1R_3)$$

$$= \sum n_i\left[\frac{k_i^2}{n_i^2}\left(1 - 2kk_i + k^2 \sum k_i^2\right)\right]$$

$$= \sum \frac{k_i^2}{n_i} - 2k \sum \frac{k_i^3}{n_i} + k^2 \sum k_i^2 \sum \frac{k_i^2}{n_i} \quad ; \tag{A56}$$

and from (A53)

$$s_{3:22} = tr(V_2R_3)^3$$

$$= \frac{N}{\sigma_e^4} - \frac{a}{\sigma_e^4} + \sum \frac{k_i^2}{n_i^2}(1 - 2kk_i) + \sum \left(\frac{k^2k_i^2}{n_i} \sum \frac{k_i^2}{n_i}\right)$$

$$= \frac{N - a}{\sigma_e^4} + \sum \frac{k_i^2}{n_i^2} - 2k \sum \frac{k_i^3}{n_i^2} + k^2\Big(\sum \frac{k_i^2}{n_i}\Big)^2 \quad . \tag{A57}$$

Using (A49) and (A50) we get

$$U_{3:1} = y'R_3V_1R_3y$$

$$= \sum_i \Big[\frac{k_i^2}{n_i^2} y_{i.}^2 + \sum_{\text{all } i'} \Big\{\frac{k_i k_{i'}}{n_i n_{i'}}\Big[k^2 \sum k_i^2 - k(k_i + k_{i'})\Big] y_{i.} y_{i'.}\Big\}\Big]$$

$$= \sum_i \Big[k_i^2 \bar{y}_{i.}^2 + \sum_{\text{all } i'} \Big\{k_i k_{i'}\Big[k^2 \sum k_i^2 - k(k_i + k_{i'})\Big]\bar{y}_{i.}\bar{y}_{i'.}\Big\}\Big]$$

$$= \sum_i \Big[k_i^2 \bar{y}_i^2 + k_i\Big\{k^2 \sum k_i^2 \sum k_i \bar{y}_{i.} - kk_i \sum k_i \bar{y}_{i.} - k \sum k_i^2 \bar{y}_{i.}\Big\}\bar{y}_{i.}\Big]$$

$$= \sum_i \Big[k_i^2 \bar{y}_{i.}^2 + k^2 \sum k_i^2 \Big(\sum k_i \bar{y}_{i.}\Big)^2 - 2k \sum k_i \bar{y}_{i.} \sum k_i^2 \bar{y}_{i.}\Big]$$

$$= \sum_i k_i^2 \Big[\bar{y}_{i.} - k \sum k_i \bar{y}_{i.}\Big]^2 \tag{A58}$$

while (A53) and (A54) yield

$$U_{3:2} = y'R_3V_2R_3y$$

$$= \frac{1}{\sigma_e^4} \sum_i \sum_j y_{ij}^2 + \sum_i \Big[\Big(\frac{-1}{n_i \sigma_e^4} + \frac{k_i^2}{n_i^3}\Big) y_{i.}^2 + \sum_{\text{all } i'} \Big\{\Big(- \frac{kk_i^2 k_{i'}}{n_i^2 n_{i'}} - \frac{kk_i k_{i'}^2}{n_i n_{i'}^2}$$

$$+ \frac{k^2 k_i k_{i'}}{n_i n_{i'}} \sum \frac{k_i^2}{n_i}\Big) y_{i.} y_{i'.}\Big\}\Big]$$

$$= \frac{1}{\sigma_e^4} \sum\sum y_{ij}^2 + \sum_i \Big[\Big(\frac{-n_i}{\sigma_e^4} + \frac{k_i^2}{n_i}\Big)\bar{y}_{i.}^2 + \sum_{\text{all } i'} \Big\{\Big(- \frac{k k_i^2 k_{i'}}{n_i} - \frac{k k_i k_{i'}^2}{n_{i'}}$$

$$+ k^2 k_i k_{i'} \sum \frac{k_i^2}{n_i}\Big)\bar{y}_{i.}\bar{y}_{i.}\Big\}\Big]$$

$$= \frac{SSE}{\sigma_e^4} + \sum_i \Big[\frac{k_i^2}{n_i}\bar{y}_{i.}^2 - \frac{k k_i^2 (\Sigma k_i \bar{y}_{i.})\bar{y}_{i.}}{n_i} - k k_i\Big(\sum \frac{k_i^2 \bar{y}_{i.}}{n_i}\Big)\bar{y}_{i.}$$

$$+ k^2 k_i \Big(\sum \frac{k_i^2}{n_i}\Big)\Big(\sum k_i \bar{y}_{i.}\Big)\bar{y}_{i.}\Big]$$

$$= \frac{SSE}{\sigma_e^4} + \sum \frac{k_i^2}{n_i}\bar{y}_{i.}^2 - 2k\Big(\sum k_i \bar{y}_{i.}\Big)\Big(\sum \frac{k_i^2 \bar{y}_{i.}}{n_i}\Big) + k^2\Big(\sum \frac{k_i^2}{n_i}\Big)\Big(\sum k_i \bar{y}_{i.}\Big)^2$$

$$= \frac{SSE}{\sigma_e^4} + \sum_i \frac{k_i^2}{n_i}\Big[\bar{y}_{i.} - k \sum k_i \bar{y}_{i.}\Big]^2 \qquad \text{(A59)}$$

where $SSE = \sum\limits_{ij}\Sigma y_{ij}^2 - \sum\limits_i n_i \bar{y}_{i.}^2$ = the customary residual or error sum of squares for the 1-way AOV.

The estimators are then

$$\hat{\sigma}_{3e}^2 = \frac{1}{\Delta_3}\,[- s_{3:12} u_{3:1} + s_{3:11} u_{3:2}] \qquad \text{(A60)}$$

and $$\hat{\sigma}^2_{3a} = \frac{1}{\Delta_3}[s_{3:22}u_{3:1} - s_{3:12}u_{3:2}] \tag{A61}$$

using (A55) through (A59) and defining $\Delta_3 = s_{3:11}s_{3:22} - s^2_{3:12}$ = the determinant of S_3 .

Using the results that $v(y'Ay) = 2tr(VA)^2$ and $cov(y'Ay, y'By) = 2\ tr(VAVB)$ when $y \sim N(\underline{\mu}, V)$, $AX = A1 = 0$, and $BX = B1 = 0$, the variances and covariance of estimators are

$$v(\hat{\sigma}^2_{3e}) = \frac{1}{\Delta_3^2}[s^2_{3:12}v(u_{3:1}) + s^2_{3:11}v(u_{3:2}) - 2s_{3:11}s_{3:12}cov(u_{3:1}, u_{3:2})] \tag{A62}$$

$$v(\hat{\sigma}^2_{3a}) = \frac{1}{\Delta_3^2}[s^2_{3:22}v(u_{3:1}) + s^2_{3:12}v(u_{3:2}) - 2s_{3:12}s_{3:22}cov(u_{3:1}, u_{3:2})]\ , \tag{A63}$$

and $$cov(\hat{\sigma}^2_{3e}, \hat{\sigma}^2_{3a}) = \frac{1}{\Delta_3^2}[-s_{3:12}s_{3:22}v(u_{3:1}) - s_{3:11}s_{3:12}v(u_{3:2}) + (s_{3:11}s_{3:22} + s^2_{3:12})cov(u_{3:1}, u_{3:2})]\ . \tag{A64}$$

However, these expressions simplify considerably. Noting that $(I - P_3)$ is idempotent and $R_3 = V^{-1}[I - P_3]$ and using (A55), (A56), and (A57), we find that under normality

$$\begin{aligned} v(u_{3:1}) &= 2tr(VR_3V_1R_3)^2 \\ &= 2tr(VV^{-1}[I - P_3]V_1R_3)^2 \\ &= 2tr([I - P_3]V_1R_3[I - P_3]V_1R_3) \\ &= 2tr(V_1R_3[I - P_3]V_1R_3(I - P_3]) \\ &= 2tr(V_1V^{-1}[I - P_3][I - P_3]V_1V^{-1}[I - P_3][I - P_3]) \end{aligned}$$

$$= 2\mathrm{tr}(V_1 V^{-1}[I - P_3]V_1 V^{-1}[I - P_3])$$

$$= 2\mathrm{tr}(V_1 R_3)^2 = 2s_{3:11} \tag{A65}$$

and by analogous steps that

$$\begin{aligned} v(u_{3:2}) &= 2\mathrm{tr}(VR_3V_2R_3)^2 \\ &= 2\mathrm{tr}(V_2R_3)^2 \\ &= 2s_{3:22} \end{aligned} \tag{A66}$$

and $$\begin{aligned} \mathrm{cov}(u_{3:1}, u_{3:2}) &= 2\mathrm{tr}(VR_3V_1R_3VR_3V_2R_3) \\ &= 2\mathrm{tr}(V_1R_3V_2R_3) = 2\mathrm{tr}(R_3V_1R_3) \\ &= 2s_{3:12} \quad . \end{aligned} \tag{A67}$$

Substituting (A65), (A66), and (A67) into (A62), (A63), and (A64) and simplifying gives

$$v(\hat{\sigma}^2_{3e}) = \frac{2s_{3:11}}{\Delta_3} \quad , \tag{A68}$$

$$v(\hat{\sigma}^2_{3a}) = \frac{2s_{3:22}}{\Delta_3} \quad , \tag{A69}$$

and $$\mathrm{cov}(\hat{\sigma}^2_{3e}, \hat{\sigma}^2_{3a}) = \frac{-2s_{3:12}}{\Delta_3} \quad . \tag{A70}$$

4. <u>MINQUE for the model with $\mu \neq 0$</u>

The elements of S_4 and U_4 and the estimators of the variance components may be obtained from the preceding MIVQUE results by setting

$\sigma_e^2 = \sigma_a^2 = 1$. A convenient notation is

$$\ell_i = \frac{n_i}{1 + n_i} \text{ and } \ell = \frac{1}{\Sigma \ell_i} \tag{A71}$$

from which

$$\ell \sum \ell_i = 1 \quad ,$$

$$1 - \ell_i = \frac{1}{1 + n_i} \quad ,$$

$$\ell_i = n_i(1 - \ell_i) \quad ,$$

and

$$n_i \ell_i = n_i - \ell_i$$

are useful identities.

From (A55) through (A59) we get

$$s_{4:11} = \sum \ell_i^2 - 2\ell \sum \ell_i^3 + \ell^2 \left(\sum \ell_i^2 \right)^2 \quad , \tag{A72}$$

$$s_{4:12} = \sum \frac{\ell_i^2}{n_i} - 2\ell \sum \frac{\ell_i^3}{n_i} + \ell^2 \sum \ell_i^2 \sum \frac{\ell_i^2}{n_i}$$

$$= \sum \ell_i(1 - \ell_i) - 2\ell \sum \ell_i^2(1 - \ell_i) + \ell^2 \sum \ell_i^2 \sum \ell_i(1 - \ell_i)$$

$$= \sum \ell_i - \sum \ell_i^2 - \ell \sum \ell_i^2 + 2\ell \sum \ell_i^3 - \ell^2 \left(\sum \ell_i^2 \right)^2$$

$$= - s_{4:11} - \ell \sum \ell_i^2 + \sum \ell_i \quad , \tag{A73}$$

$$s_{4:22} = N - a + \sum \frac{\ell_i^2}{n_i^2} - 2\ell \sum \frac{\ell_i^3}{n_i^2} + \ell^2\Big(\sum \frac{\ell_i^2}{n_i}\Big)^2$$

$$= N - a + \sum (1 - \ell_i)^2 - 2\ell \sum \ell_i(1 - \ell_i)^2 + \ell^2 \Big(\sum \ell_i[1 - \ell_i]\Big)^2$$

$$= N - a + a - 2\sum \ell_i + \sum \ell_i^2 - 2 + 4\ell \sum \ell_i^2 - 2\ell \sum \ell_i^3 + 1$$

$$- 2\ell \sum \ell_i^2 + \ell^2\Big(\sum \ell_i^2\Big)^2$$

$$= s_{4:11} + N - 1 - 2\sum \ell_i + 2\ell \sum \ell_i^2 \quad , \tag{A74}$$

$$u_{4:1} = \sum \ell_i^2 \left[\bar{y}_{i\cdot} - \ell \sum \ell_i \bar{y}_{i\cdot}\right]^2 , \tag{A75}$$

and $$u_{4:2} = SSE + \sum \frac{\ell_i^2}{n_i} \left[\bar{y}_{i\cdot} - \ell \sum \ell_i \bar{y}_{i\cdot}\right]^2$$

$$= SSE + \sum \ell_i(1 - \ell_i)\left[\bar{y}_{i\cdot} - \ell \sum \ell_i \bar{y}_{i\cdot}\right]^2$$

$$= -u_{4:1} + SSE + \sum \ell_i\left[\bar{y}_{i\cdot} - \ell \sum \ell_i \bar{y}_{i\cdot}\right]^2 \quad . \tag{A76}$$

The estimators are

$$\hat{\sigma}_{4e}^2 = \frac{1}{\Delta_4} \left[-s_{4:12}u_{4:1} + s_{4:11}u_{4:2}\right] \tag{A77}$$

and $$\hat{\sigma}_{4a}^2 = \frac{1}{\Delta_4} \left[s_{4:22}u_{4:1} - s_{4:12}u_{4:2}\right] \tag{A78}$$

using (A72) through (A76) and defining $\Delta_4 = s_{4:11}s_{4:22} - s_{4:12}^2$ the determinant of S^4 . Δ_4 can also be found as $\Delta_4 = (N - 1)s_{4:11} - [\ell\Sigma\ell_i^2 - \ell^{-1}]^2$ but this form is not very useful since one would still calculate $s_{4:11}$, $s_{4:12}$, and $s_{4:22}$ for (A77) and (A78).

The variances and covariance of the estimators are given by

$$v(\hat{\sigma}^2_{4e}) = \frac{1}{\Delta_4^2}[s^2_{4:12}v(u_{4:1}) + s^2_{4:11}v(u_{4:2}) - 2s_{4:11}s_{4:12}\mathrm{cov}(u_{4:1},u_{4:2})] , \tag{A79}$$

$$v(\hat{\sigma}^2_{4a}) = \frac{1}{\Delta_4^2}[s^2_{4:22}v(u_{4:1}) + s^2_{4:12}v(u_{4:2}) - 2s_{4:12}s_{4:22}\mathrm{cov}(u_{4:1},u_{4:2})] , \tag{A80}$$

and $$\mathrm{cov}(\hat{\sigma}^2_{4e},\hat{\sigma}^2_{4a}) = \frac{1}{\Delta_4^2}[-s_{4:12}s_{4:22}v(u_{4:1}) - s_{4:11}s_{4:12}v(u_{4:2}) + (s_{4:11}s_{4:22} + s^2_{4:12})\mathrm{cov}(u_{4:1},u_{4:2})] . \tag{A81}$$

Expressions for $v(u_{4:1})$, $v(u_{4:2})$, and $\mathrm{cov}(u_{4:1},u_{4:2})$ are not obtainable from the MIVQUE case, so are derived below.

We can get $R_4V_1R_4$ and $R_4V_2R_4$ from the analogous MIVQUE expressions by setting $\sigma_e^2 = \sigma_a^2 = 1$. From (A49) and (A50) $R_4V_1R_4$ has its i^{th} diagonal block

$$\frac{\ell_i^2}{n_i^2}\left(1 - 2\ell\ell_i + \ell^2\sum\ell_i^2\right)J_{n_i} = (1 - \ell_i)^2\left(1 - 2\ell\ell_i + \ell^2\sum\ell_i^2\right)J_{n_i} \tag{A82}$$

and its i,i'^{th} off-diagonal block

$$\frac{\ell_i\ell_{i'}\left(\ell^2\sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]\right)}{n_i n_{i'}} J_{n_i \times n_{i'}}$$

$$= (1 - \ell_i)(1 - \ell_{i'})(\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}])\, J_{n_i \times n_{i'}} \quad . \tag{A83}$$

From (A53) and (A54) $R_4V_2R_4$ has its i^{th} diagonal block

$$I_{n_i} + \left[\frac{-1}{n_i} + \frac{\ell_i^2(1 - 2\ell\ell_i)}{n_i^3} + \frac{\ell^2\ell_i^2}{n_i^2}\sum \frac{\ell_i^2}{n_i}\right] J_{n_i}$$

$$= I_{n_i} + \left[\frac{-1 + (1 - \ell_i)^2(1 - 2\ell\ell_i) + \ell^2\ell_i(1 - \ell_i)\Sigma\ell_i(1 - \ell_i)}{n_i}\right] J_{n_i}$$

$$= I_{n_i} + \left[\frac{-1 + (1-\ell_i)(1-2\ell\ell_i) + 2\ell\ell_i(1-\ell_i) + \ell_i(1-\ell_i)(-1-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2)}{n_i}\right] J_{n_i}$$

$$= I_{n_i} + \left[\frac{-\ell_i + \ell_i(1 - \ell_i)(-1 - \ell + 2\ell\ell_i - \ell^2\Sigma\ell_i^2)}{n_i}\right] J_{n_i}$$

$$= I_{n_i} + \left[-(1 - \ell_i) + (1 - \ell_i)^2(-1 - \ell + 2\ell\ell_i - \ell^2\Sigma\ell_i^2)\right] J_{n_i} \tag{A84}$$

and its i,i'^{th} off-diagonal block

$$\left[- \frac{\ell\ell_i^2\ell_{i'}}{n_i^2 n_{i'}} - \frac{\ell\ell_i\ell_{i'}^2}{n_i n_{i'}^2} + \frac{\ell^2\ell_i\ell_{i'}}{n_i n_{i'}}\sum \frac{\ell_i^2}{n_i}\right] J_{n_i \times n_{i'}}$$

$$= (1 - \ell_i)(1 - \ell_{i'})\left[-\ell(1 - \ell_i) - \ell(1 - \ell_{i'}) + \ell^2 \sum \ell_i(1 - \ell_i)\right] J_{n_i \times n_{i'}}$$

$$= (1 - \ell_i)(1 - \ell_{i'})\left[-\ell + \ell(\ell_i + \ell_{i'}) - \ell^2 \sum \ell_i^2\right] J_{n_i \times n_{i'}} \quad . \tag{A85}$$

Now $v(u_{4:1}) = 2\mathrm{tr}(VR_4V_1R_4)^2$ where from (A82) and (A83) and k_i defined by (A43) $VR_4V_1R_4$ has its i^{th} diagonal block

$$\left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right]\left[(1 - \ell_i)^2(1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i) J_{n_i}\right]$$

$$= \left(\sigma_e^2 + n_i\sigma_a^2\right)\left(1 - \ell_i\right)^2\left(1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i\right) J_{n_i}$$

$$= \frac{n_i}{k_i}\left(1 - \ell_i\right)^2\left(1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i\right) J_{n_i}$$

$$= \frac{\ell_i(1 - \ell_i)}{k_i}\left(1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i\right) J_{n_i} \tag{A86}$$

and its i,i'^{th} off-diagonal block

$$\left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right]\left[(1 - \ell_i)(1 - \ell_{i'})(\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i \times n_{i'}}\right]$$

$$= \frac{n_i}{k_i}(1 - \ell_i)(1 - \ell_{i'})(\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i \times n_{i'}}$$

$$= \frac{\ell_i(1 - \ell_{i'})}{k_i}(\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]) J_{n_i \times n_{i'}} \quad . \tag{A87}$$

From (A86) and (A87) $(VR_4V_1R_4)^2$ has its i^{th} diagonal block

$$\left[\frac{\ell_i^3(1 - \ell_i)}{k_i^2}\left(1 + \ell^2 \sum \ell_i^2 - 2\ell\ell_i\right)^2 + \sum_{i' \neq i} \frac{\ell_i\ell_{i'}^2(1-\ell_i)}{k_ik_{i'}}\left(\ell^2 \sum \ell_i^2 - \ell[\ell_i + \ell_{i'}]\right)^2\right] J_{n_i}$$

$$= \left[\frac{\ell_i^3(1 - \ell_i)}{k_i^2}\left(1 + 2\ell^2 \sum \ell_i^2 - 4\ell\ell_i\right)\right.$$

$$+ \frac{\ell_i(1-\ell_i)}{k_i} \sum_{\text{all } i'} \frac{\ell_{i'}^2}{k_{i'}} \Big\{\ell^4\Big(\sum \ell_i^2\Big)^2 + \ell^2\ell_i^2 + \ell^2\ell_{i'}^2$$

$$- 2\ell^3\ell_i \sum \ell_i^2 - 2\ell^3\ell_{i'} \sum \ell_i^2 + 2\ell^2\ell_i\ell_{i'}\Big\}\Big] J_{n_i}$$

$$= \Big[\frac{\ell_i^3(1-\ell_i)}{k_i^2}\Big(1 + 2\ell^2 \sum \ell_i^2 - 4\ell\ell_i\Big)$$

$$+ \frac{\ell_i(1-\ell_i)}{k_i}\Big\{\ell^4\Big(\sum \ell_i^2\Big)^2 \sum \frac{\ell_i^2}{k_i} + \ell^2\ell_i^2 \sum \frac{\ell_i^2}{k_i} + \ell^2 \sum \frac{\ell_i^4}{k_i}$$

$$- 2\ell^3\ell_i \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} - 2\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i} + 2\ell^2\ell_i \sum \frac{\ell_i^3}{k_i}\Big\}\Big] J_{n_i} \quad . \qquad \text{(A88)}$$

Then $v(u_{4:1}) = 2\mathrm{tr}(VR_4V_1R^4)^2$

$$= 2\Big[\sum \frac{\ell_i^4}{k_i^2} + 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^4}{k_i^2} - 4\ell \sum \frac{\ell_i^5}{k_i^2} + \ell^4\Big(\sum \ell_i^2\Big)^2\Big(\sum \frac{\ell_i^2}{k_i}\Big)^2$$

$$+ 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^4}{k_i} - 4\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} + 2\ell^2\Big(\sum \frac{\ell_i^3}{k_i}\Big)^2\Big] \quad . \qquad \text{(A89)}$$

Similarly, $v(u_{4:2}) = 2\mathrm{tr}(VR_4V_2R_4)^2$ where from (A84) and (A85) $VR_4V_2R_4$ has its i^{th} diagonal block

$$\Big[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\Big]\Big[I_{n_i} + \Big\{-(1-\ell_i) + (1-\ell_i)^2(-1-\ell+2\ell\ell_i - \ell^2 \sum \ell_i^2)\Big\} J_{n_i}\Big]$$

$$= \sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \frac{n_i}{k_i}\Big\{-(1-\ell_i) + (1-\ell_i)^2(-1-\ell+2\ell\ell_i - \ell^2 \sum \ell_i^2)\Big\} J_{n_i}$$

$$= \sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \frac{1}{k_i}\Big\{-\ell_i + \ell_i(1-\ell_i)(-1-\ell+2\ell\ell_i - \ell^2 \sum \ell_i^2)\Big\} J_{n_i} \qquad \text{(A90)}$$

and its i,i'^{th} off-diagonal block

$$\left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right]\left[(1 - \ell_i)(1 - \ell_{i'})(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)\ J_{n_i \times n_{i'}}\right]$$

$$= \frac{n_i}{k_i}\ (1 - \ell_i)(1 - \ell_{i'})(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)\ J_{n_i \times n_{i'}}$$

$$= \frac{\ell_i}{k_i}\ (1 - \ell_{i'})(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)\ J_{n_i \times n_{i'}}\ . \tag{A91}$$

So from (A90) and (A91) $(VR_4V_2R_4)^2$ has its i^{th} diagonal block

$$\left[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \left\{\frac{-\ell_i - \ell_i(1 - \ell_i)}{k_i} + \frac{\ell_i(1 - \ell_i)}{k_i}\left(-\ell + 2\ell\ell_i - \ell^2 \sum \ell_i^2\right)\right\} J_{n_i}\right]^2$$

$$+ \sum_{i' \neq i} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}}\ (1 - \ell_i)(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)^2\ J_{n_i}$$

$$= \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right)^2 + \left[\frac{n_i(-\ell_i - \ell_i[1 - \ell_i])^2}{k_i^2}\right.$$

$$+ 2\frac{n_i}{k_i}\left\{\frac{-\ell_i + \ell_i(1 - \ell_i)(-1 - \ell + 2\ell\ell_i - \ell^2\Sigma\ell_i^2)}{k_i}\right\}$$

$$+ 2\ \frac{\ell_i^2(-\ell_i - \ell_i[1 - \ell_i])}{k_i^2}\left(-\ell + 2\ell\ell_i - \ell^2 \sum \ell_i^2\right)$$

$$\left. + \sum_{\text{all } i'} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}}\ (1 - \ell_i)(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)^2\right] J_{n_i}$$

$$= \left(\sigma_e^2 I_{n_i} + \sigma^2 J_{n_i}\right)^2$$

$$+ \left[\frac{n_i\ell_i^2+2\ell_i^3+\ell_i^3(1-\ell_i)-2n_i\ell_i+2\ell_i^2(-1-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2)+2\ell_i^3(-1-[1-\ell_i])(-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2)}{k_i^2}\right.$$

$$+ \sum_{\text{all } i'} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}} (1 - \ell_i)(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)^2 \Big] J_{n_i}$$

$$= \left(\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i}\right)^2$$

$$+ \Big[\frac{n_i \ell_i^2 + 3\ell_i^3 - \ell_i^4 - 2n_i \ell_i - 2\ell_i^2 + 2\ell_i^2 (1 - \ell_i)^2 (-\ell + 2\ell\ell_i - \ell^2 \Sigma \ell_i^2)}{k_i^2}$$

$$+ \sum_{\text{all } i'} \frac{\ell_i \ell_{i'}^2}{k_i k_{i'}} (1 - \ell_i)(-\ell + \ell[\ell_i + \ell_{i'}] - \ell^2 \sum \ell_i^2)^2 \Big] J_{n_i}$$

$$= \sigma_e^4 I_{n_i} + 2\sigma_e^2 \sigma_a^2 J_{n_i} + n_i \sigma_a^4 J_{n_i}$$

$$+ \Big[\frac{-n_i + \ell_i (1 - \ell_i)^2 (1 - \ell_i - 2\ell\ell_i + 4\ell\ell_i^2 - 2\ell^2 \ell_i \Sigma \ell_i^2)}{k_i^2}$$

$$+ \frac{\ell_i (1 - \ell_i)}{k_i} \Big\{\ell^2 \sum \frac{\ell_i^2}{k_i} + \ell^2 \ell_i^2 \sum \frac{\ell_i^2}{k_i} + \ell^2 \sum \frac{\ell_i^4}{k_i} + \ell^4 \Big(\sum \ell_i^2\Big)^2 \sum \frac{\ell_i^2}{k_i}$$

$$- 2\ell^2 \ell_i \sum \frac{\ell_i^2}{k_i} - 2\ell^2 \sum \frac{\ell_i^3}{k_i} + 2\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} + 2\ell^2 \ell_i \sum \frac{\ell_i^3}{k_i}$$

$$- 2\ell^3 \ell_i \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} - 2\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i}\Big\}\Big] J_{n_i} \qquad \text{(A92)}$$

using the identity

$$n_i \ell_i^2 + 3\ell_i^3 - \ell_i^4 - 2n_i \ell_i - 2\ell_i^2 = -n_i + \ell_i (1 - \ell_i)^3$$

to obtain the last equality.

From (A92) and then (A89) and (A95) we get

$$v(u_{4:2}) = 2\mathrm{tr}(VR_4 V_a R_4)^2$$

$$
\begin{aligned}
&= 2\Big[N\sigma_e^4 + 2N\sigma_e^2\sigma_a^2 + \sum n_i^2\sigma_a^4 - \sum \frac{n_i^2}{k_i^2} \\
&\quad + \sum \frac{\ell_i^2}{k_i^2} - 2\sum \frac{\ell_i^3}{k_i^2} - 2\ell \sum \frac{\ell_i^3}{k_i^2} + 6\ell \sum \frac{\ell_i^4}{k_i^2} \\
&\quad - 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i^2} + \sum \frac{\ell_i^4}{k_i^2} - 4\ell \sum \frac{\ell_i^5}{k_i^2} \\
&\quad + 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^4}{k_i^2} + \ell^2\Big(\sum \frac{\ell_i^2}{k_i}\Big)^2 \\
&\quad + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^4}{k_i} + \ell^4\Big(\sum \ell_i^2\Big)^2\Big(\sum \frac{\ell_i^2}{k_i}\Big)^2 \\
&\quad - 4\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} + 2\ell^3 \sum \ell_i^2\Big(\sum \frac{\ell_i^2}{k_i}\Big)^2 \\
&\quad + 2\ell^2\Big(\sum \frac{\ell_i^3}{k_i}\Big)^2 - 4\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i}\Big] \\
&= 2\Big[-\tfrac{1}{2}\, v(u_{4:1}) - \operatorname{cov}(u_{4:1}, u_{4:2}) + N\sigma_e^4 \\
&\quad + 2N\sigma_e^2\sigma_a^2 + \sum n_i^2\sigma_a^4 - \sum \frac{n_i^2}{k_i^2} + \sum \frac{\ell_i^2}{k_i^2} \\
&\quad - 2\ell \sum \frac{\ell_i^3}{k_i^2} + \ell^2\Big(\sum \frac{\ell_i^2}{k_i}\Big)^2\Big] \;. \qquad \text{(A93)}
\end{aligned}
$$

Finally, $\operatorname{cov}(u_{4:1}, u_{4:2}) = 2\operatorname{tr}(VR_4V_1R_4VR_4V_2R_4)$ where from (A86), (A87), (A90), and (A91) $VR_4V_1R_4VR_4V_2R_4$ has its i^{th} diagonal block

$$
\Big[\frac{-\ell_i(1-\ell_i)}{k_i}\left(1+\ell^2\Sigma\ell_i^2-2\ell\ell_i\right)J_{n_i}\Big]\Big[\sigma_e^2 I_{n_i} + \sigma_a^2 J_{n_i} + \frac{1}{k_i}\Big\{-\ell_i+\ell_i(1-\ell_i)(-1-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2)\Big\}J_{n_i}\Big]
$$

$$+ \sum_{i'\neq i} \left[\frac{\ell_i}{k_i}(1-\ell_{i'})(\ell^2\Sigma\ell_i^2-\ell[\ell_i+\ell_{i'}])\, J_{n_i \times n_{i'}}\right]\left[\frac{\ell_{i'}}{k_{i'}}(1-\ell_i)(-\ell+\ell[\ell_i+\ell_{i'}]\right.$$

$$\left. -\ell^2\Sigma\ell_i^2) J_{n_{i'} \times n_i}\right]$$

$$= \left[\frac{\ell_i^2}{k_i^2}(1+\ell^2\Sigma\ell_i^2-2\ell\ell_i)+\frac{\ell_i^2}{k_i^2}(1+\ell^2\Sigma\ell_i^2-2\ell\ell_i)\left\{-\ell_i+\ell_i(1-\ell_i)(-1-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2\right\}\right.$$

$$\left. + \sum_{i'\neq i} \frac{\ell_i\ell_{i'}^2}{k_i k_{i'}}(1-\ell_i)(\ell^2\Sigma\ell_i^2-\ell[\ell_i+\ell_{i'}])(-\ell+\ell[\ell_i+\ell_{i'}]-\ell^2\Sigma\ell_i^2)\right] J_{n_i}$$

$$= \left[\frac{\ell_i^2}{k_i^2}\left\{(1+\ell^2\Sigma\ell_i^2-2\ell\ell_i)+(-\ell_i+\ell_i[1-\ell_i][-1-\ell+2\ell\ell_i-\ell^2\Sigma\ell_i^2])\right.\right.$$

$$\left. + (\ell^2\Sigma\ell_i^2-2\ell\ell_i)(-\ell_i-\ell_i[1-\ell_i])\right\} + \sum_{\text{all } i'} \frac{\ell_i\ell_{i'}^2}{k_i k_{i'}}(1-\ell_i)(\ell^2\Sigma\ell_i^2-\ell[\ell_i+\ell_{i'}])(-\ell$$

$$\left. +\ell[\ell_i+\ell_{i'}]-\ell^2\Sigma\ell_i^2)\right] J_{n_i}$$

$$= \left[\frac{\ell_i^2}{k_i^2}\left\{(1-\ell_i)(1-\ell_i-\ell\ell_i+[\ell^2\Sigma\ell_i^2-2\ell\ell_i][1-2\ell_i])\right\}\right.$$

$$+ \frac{\ell_i(1-\ell_i)}{k_i}\left\{-\ell^3\Sigma\ell_i^2\Sigma\frac{\ell_i^2}{k_i} + 2\ell^3\ell_i\ \Sigma\ell_i^2\ \ \Sigma\frac{\ell_i^2}{k_i} + 2\ell^3\Sigma\ell_i^2\Sigma\frac{\ell_i^3}{k_i} - \ell^4(\Sigma\ell_i^2)^2\Sigma\frac{\ell_i^2}{k_i}\right.$$

$$\left.\left. + \ell^2\ell_i\Sigma\frac{\ell_i^2}{k_i} - \ell^2\ell_i^2\Sigma\frac{\ell_i^2}{k_i} - 2\ell^2\ell_i\Sigma\frac{\ell_i^3}{k_i} + \ell^2\Sigma\frac{\ell_i^3}{k_i} - \ell^2\Sigma\frac{\ell_i^4}{k_i}\right\}\right]J_{n_i} \quad . \tag{A94}$$

From (A94) and then (A89)

$$\text{cov}(u_{4:1},u_{4:2}) = 2\ \text{tr}(VR_4V_1R_4VR_4V_2R_4)$$

$$= 2\left[\sum\frac{\ell_i^3}{k_i^2} - \sum\frac{\ell_i^4}{k_i^2} - 3\ell\sum\frac{\ell_i^4}{k_i^2} + \ell^2\sum\ell_i^2\sum\frac{\ell_i^3}{k_i^2}\right.$$

$$- 2\ell^2 \sum \ell_i^2 \sum \frac{\ell_i^4}{k_i^2} + 4\ell \sum \frac{\ell_i^5}{k_i^2} - \ell^3 \sum \ell_i^2 \Big(\sum \frac{\ell_i^2}{k_i}\Big)^2$$

$$+ 4\ell^3 \sum \ell_i^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} - \ell^4 \Big(\sum \ell_i^2\Big)^2 \Big(\sum \frac{\ell_i^2}{k_i}\Big)^2$$

$$+ 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} - 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^4}{k_i} - 2\ell^2 \Big(\sum \frac{\ell_i^3}{k_i}\Big)^2 \Big]$$

$$= 2\Big[-\tfrac{1}{2} v(u_{4:1}) + \sum \frac{\ell_i^3}{k_i^2} - 3\ell \sum \frac{\ell_i^4}{k_i^2} + \ell^2 \sum \ell_i^2 \sum \frac{\ell_i^3}{k_i^2}$$

$$- \ell^3 \sum \ell_i^2 \Big(\sum \frac{\ell_i^2}{k_i}\Big)^2 + 2\ell^2 \sum \frac{\ell_i^2}{k_i} \sum \frac{\ell_i^3}{k_i} \Big] \quad . \tag{A95}$$

II. Simplification under balance of the MINQUE and MIVQUE estimators and their variances and covariances

Under the condition of balance, i.e. $n_1 = n_2 = \cdots = n_a = n$, we can show that the estimators for MINQUE and MIVQUE reduce to the usual AOV estimators. The variances and covariances simplify correspondingly.

1. MIVQUE with $\mu \equiv 0$

When $n_1 = n_2 = \cdots = n_a = n$ and therefore for q_i defined in (A7) $q_1 = q_2 = \cdots = q_a = \bar{q}.$, (A8) through (A12) give

$$s_{1:11} = \frac{an^2\bar{q}_.^2}{\sigma_e^4} , \tag{A96}$$

$$s_{1:12} = \frac{an\bar{q}_.^2}{\sigma_e^4} , \tag{A97}$$

$$s_{1:22} = \frac{a\bar{q}_.^2 + N - a}{\sigma_e^4} , \tag{A98}$$

$$u_{1:1} = \frac{\bar{q}_.^2 \sum y_{i.}^2}{\sigma_e^4} , \tag{A99}$$

and

$$u_{1:2} = \frac{\sum\sum y_{ij}^2 - \sum \frac{y_{i.}^2}{n} + \bar{q}_.^2 \sum \frac{y_{i.}^2}{n}}{\sigma_e^4}$$

$$\doteq \frac{SSE + \frac{\bar{q}_.^2}{n} \sum y_{i.}^2}{\sigma_e^4} . \tag{A100}$$

Then from (A96), (A97), and (A98)

$$\Delta_1 = s_{1:11}s_{1:22} - s_{1:12}^2$$

$$= \frac{(N - a)s_{1:11}}{\sigma_e^4} . \tag{A101}$$

Substituting (A96) through (A101) into (A13) and (A14) gives

$$\hat{\sigma}^2_{1e} = \frac{\sigma_e^4}{(N-a)s_{1:11}}\left[-\left(\frac{an\bar{q}_.^2}{\sigma_e^4}\right)\left(\frac{\bar{q}_.^2\Sigma y_{i.}^2}{\sigma_e^4}\right) + \left(\frac{an^2\bar{q}_.^2}{\sigma_e^4}\right)\left(\frac{SSE + \frac{\bar{q}_.^2}{n}\Sigma y_{i.}^2}{\sigma_e^4}\right)\right]$$

$$= \frac{s_{1:11}SSE}{(N-a)s_{1:11}} = \frac{SSE}{N-a} = MSE \tag{A102}$$

and

$$\hat{\sigma}^2_{1a} = \frac{\sigma_e^4}{(N-a)s_{1:11}}\left[\left(\frac{a\bar{q}_.^2 + N - a}{\sigma_e^4}\right)\left(\frac{\bar{q}_.^2\Sigma y_{i.}^2}{\sigma_e^4}\right) - \left(\frac{an\bar{q}_.^2}{\sigma_e^4}\right)\left(\frac{SSE + \frac{\bar{q}_.^2}{n}\Sigma y_{i.}^2}{\sigma_e^4}\right)\right]$$

$$= \frac{\sigma_e^4}{(N-a)s_{1:11}}\left[\frac{(N-a)\bar{q}_.^2\Sigma y_{i.}^2}{\sigma_e^8} - \frac{an\bar{q}_.^2 SSE}{\sigma_e^8}\right]$$

$$= \frac{\sigma_e^4}{(N-a)s_{1:11}}\left[\frac{(N-a)an^2\bar{q}_.^2\Sigma \frac{y_{i.}^2}{n}}{an\sigma_e^8} - \frac{an^2\bar{q}_.^2 SSE}{n\sigma_e^8}\right]$$

$$= \frac{\sigma_e^4}{(N-a)s_{1:11}}\left[\frac{(N-a)s_{1:11}SSA}{an\sigma_e^4} - \frac{s_{1:11}SSE}{n\sigma_e^4}\right]$$

$$= \frac{1}{n}\left[\frac{SSA}{a} - \frac{SSE}{N-a}\right]$$

$$= \frac{1}{n}[MSA - MSE] \tag{A103}$$

where SSE and SSA are respectively the error and treatment sums of squares for the 1-way AOV with $\mu = 0$, MSE and MSA are the corresponding mean squares, and (A102) and (A103) are the customary AOV estimators of the variance components.

Writing (A23), (A24), and (A25) in terms of (A96) through (A101) gives

$$v(\hat{\sigma}^2_{1e}) = \frac{2s_{1:11}}{\Delta_1}$$

$$= \frac{2\sigma_e^4 s_{1:11}}{(N-a)s_{1:11}} = \frac{2\sigma_e^4}{N-a} , \qquad (A104)$$

$$v(\hat{\sigma}_{1a}^2) = \frac{2s_{1:22}}{\Delta_1}$$

$$= \frac{2\sigma_e^4 s_{1:22}}{(N-a)s_{1:11}}$$

$$= \frac{2\sigma_e^4[a\bar{q}_.^2 + N - a]}{(N-a)an^2\bar{q}_.^2}$$

$$= \frac{2\sigma_e^4}{n^2(N-a)} + \frac{2\sigma_e^4}{an^2\bar{q}_.^2}$$

$$= \frac{2\sigma_e^4}{n^2(N-a)} + \frac{2(\sigma_e^2 + n\sigma_a^2)^2}{nN} , \qquad (A105)$$

since $\bar{q}_. = \dfrac{\sigma_e^2}{\sigma_e^2 + n\sigma_a^2}$ from (A7) and $N = an$, and

$$\mathrm{cov}(\hat{\sigma}_{1e}^2, \hat{\sigma}_{1a}^2) = \frac{-2s_{1:12}}{\Delta_1}$$

$$= \frac{-2\sigma_e^4 s_{1:12}}{(N-a)s_{1:11}}$$

$$= \frac{-2\sigma_e^4 an\bar{q}_.^2}{(N-a)an^2\bar{q}_.^2}$$

$$= \frac{-2\sigma_e^4}{n(N-a)} . \qquad (A106)$$

2. <u>MINQUE with $\mu \equiv 0$</u>

Since the MINQUE estimators can be regarded as a special case of MIVQUE, the simplifications completed at (A102) and (A103) apply to MINQUE. We next show that under balance the variances and covariance of the MINQUE estimators reduce to (A104), (A105) and (A106).

Under balance $w_1 = w_2 = \cdots = w_a = \bar{w}_.$, using w_i of (A27). Then setting $\sigma_e^2 = \sigma_a^2 = 1$ in (A96), (A97), (A98), and (A101)

$$s_{2:11} = an^2\bar{w}_.^2 \ , \tag{A107}$$

$$s_{2:12} = an\bar{w}_.^2 \ , \tag{A108}$$

$$s_{2:22} = a\bar{w}_.^2 + N - a \ , \tag{A109}$$

and

$$\Delta_2 = s_{2:11}s_{2:22} - s_{2:12}^2$$

$$= (N - a)s_{2:11} \ . \tag{A110}$$

From (A39), (A41), and (A42)

$$v(u_{2:1}) = \frac{2an^2\bar{w}_.^4\sigma_e^4}{\bar{q}_.^2} \ , \tag{A111}$$

$$v(u_{2:2}) = 2\sigma_e^4\left[\frac{a\bar{w}_.^4}{\bar{q}_.^2} + N - a\right] \ , \tag{A112}$$

and

$$\mathrm{cov}(u_{2:1},u_{2:2}) = \frac{2an\bar{w}_.^4\sigma_e^4}{\bar{q}_.^2} \ . \tag{A113}$$

Now substituting (A107) through (A113) into (A35), (A36), and (A37)

$$v(\hat{\sigma}^2_{2e}) = \frac{2\sigma^4_e}{\Delta^2_2}\Big[\left(an\bar{w}^2_.\right)^2\left(\frac{an^2\bar{w}^4_.}{\bar{q}^2_.}\right) + \left(an^2\bar{w}^2_.\right)^2\left(\frac{a\bar{w}^4_.}{\bar{q}^2_.} + N - a\right)$$

$$-2\left(an^2\bar{w}^2_.\right)\left(an\bar{w}^2_.\right)\left(\frac{an\bar{w}^4_.}{\bar{q}^2_.}\right)\Big]$$

$$= \frac{2\sigma^4_e(N - a)(an^2\bar{w}^2_.)^2}{(N - a)^2(an^2\bar{w}^2_.)^2} = \frac{2\sigma^4_e}{N - a}\,, \tag{A114}$$

$$v(\hat{\sigma}^2_{2a}) = \frac{2\sigma^4_e}{\Delta^2_2}\Big[\left(a\bar{w}^2_. + N - a\right)^2\left(\frac{an^2\bar{w}^4_.}{\bar{q}^2_.}\right) + \left(an\bar{w}^2_.\right)^2\left(\frac{a\bar{w}^4_.}{\bar{q}^2_.} + N - a\right)$$

$$-2\left(an\bar{w}^2_.\right)\left(a\bar{w}^2_. + N - a\right)\left(\frac{an\bar{w}^4_.}{\bar{q}^2_.}\right)\Big]$$

$$= \frac{2\sigma^4_e\Big[(N - a)^2\,\frac{an^2\bar{w}^4_.}{\bar{q}^2_.} + (N - a)a^2n^2\bar{w}^4_.\Big]}{(N - a)^2(an^2\bar{w}^2_.)^2}$$

$$= \frac{2\sigma^4_e}{an^2\bar{q}^2_.} + \frac{2\sigma^4_e}{n^2(N - a)}$$

$$= \frac{2(\sigma^2_e + n\sigma^2_a)^2}{nN} + \frac{2\sigma^4_e}{n^2(N - a)}\,, \tag{A115}$$

recalling that $\bar{q}_. = \dfrac{\sigma^2_e}{\sigma^2_e + n\sigma^2_a}$ and $N = an$, and

$$\mathrm{cov}(\hat{\sigma}^2_{2e},\hat{\sigma}^2_{2a}) = \frac{2\sigma^4_e}{\Delta^2_2}\Big[-\left(an\bar{w}^2_.\right)\left(a\bar{w}^2_. + N - a\right)\left(\frac{an^2\bar{w}^4_.}{\bar{q}^2_.}\right)$$

$$- \left(an^2\bar{w}^2_.\right)\left(an\bar{w}^2_.\right)\left(\frac{a\bar{w}^4_.}{\bar{q}^2_.} + N - a\right)$$

$$+ \left\{\left(an^2\bar{w}^2_.\right)\left(a\bar{w}^2_. + N - a\right) + \left(an\bar{w}^2_.\right)^2\right\}\left(\frac{an\bar{w}^4_.}{\bar{q}^2_.}\right)\Big]$$

$$= \frac{2\sigma_e^4\left[-(N - a)a^2n^3\bar{w}_\cdot^4\right]}{(N - a)^2(an^2\bar{w}_\cdot^2)^2}$$

$$= \frac{-2\sigma_e^4}{n(N - a)} \quad . \qquad (116)$$

Note that (A114), (A115) and (A116) equal respectively (A104), (A105), and (A106) and are the variances and covariance of the usual AOV estimators when $\mu \equiv 0$.

3. MIVQUE with $\mu \not\equiv 0$

Under balance (A43) gives $k_1 = k_2 = \cdots = k_a = \bar{k}_.$ and $k = \frac{1}{a\bar{k}_.}$. Then from (A55) through (A59) we have

$$s_{3:11} = a\bar{k}_\cdot^2 - 2\bar{k}_\cdot^2 + \bar{k}_\cdot^2 = (a - 1)\bar{k}_\cdot^2 \ , \qquad (A117)$$

$$s_{3:12} = \frac{a\bar{k}_\cdot^2}{n} - \frac{2\bar{k}_\cdot^2}{n} + \frac{\bar{k}_\cdot^2}{n} = \frac{(a - 1)\bar{k}_\cdot^2}{n} \ , \qquad (A118)$$

$$s_{3:22} = \frac{N - a}{\sigma_e^4} + \frac{a\bar{k}_\cdot^2}{n^2} - \frac{2\bar{k}_\cdot^2}{n^2} + \frac{\bar{k}_\cdot^2}{n^2}$$

$$= \frac{N - a}{\sigma_e^4} + (a - 1)\frac{\bar{k}_\cdot^2}{n^2} \ , \qquad (A119)$$

$$\Delta_3 = s_{3:11}s_{3:22} - s_{3:12}^2$$

$$= \frac{(N - a)s_{3:11}}{\sigma_e^4} \ , \qquad (A120)$$

$$u_{3:1} = \bar{k}_\cdot^2 \sum \left(\bar{y}_{i\cdot} - \frac{1}{a\bar{k}_\cdot}\bar{k}_\cdot \sum \bar{y}_{i\cdot}\right)^2$$

$$= \bar{k}_{\cdot}^2 \sum \left(\bar{y}_{i\cdot} - \frac{a\bar{y}_{\cdot\cdot}}{a}\right)^2$$

$$= \bar{k}_{\cdot}^2 \sum \left(\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot}\right)^2$$

$$= \frac{\bar{k}_{\cdot}^2 SSA_m}{n} \;, \qquad \text{(A121)}$$

and by similar steps

$$u_{3:2} = \frac{SSE}{\sigma_e^4} + \frac{\bar{k}_{\cdot}^2 SSA_m}{n^2} \;, \qquad \text{(A122)}$$

where SSE and SSA_m are respectively the usual error and treatment sums of squares and MSE and MSA_m are the corresponding mean squares for the 1-way AOV with mean μ .

Substituting (A117) through (A122) into (A60) and (A61) gets

$$\hat{\sigma}_{3e}^2 = \frac{\sigma_e^4}{(N - a)s_{3:11}}\left[-\left(\frac{(a - 1)\bar{k}_{\cdot}^2}{n}\right)\left(\frac{\bar{k}_{\cdot}^2 SSA_m}{n}\right)\right.$$

$$\left. + \left((a - 1)\bar{k}_{\cdot}^2\right)\left(\frac{SSE}{\sigma_e^4} + \frac{\bar{k}_{\cdot}^2 SSA_m}{n^2}\right)\right]$$

$$= \frac{\sigma_e^4 (a - 1)\bar{k}_{\cdot}^2 SSE}{(N - a)\sigma_e^4 s_{3:11}} = \frac{SSE}{N - a} = MSE \qquad \text{(A123)}$$

and

$$\hat{\sigma}_{3a}^2 = \frac{\sigma_e^4}{(N - a)s_{3:11}}\left[\left(\frac{N - a}{\sigma_e^4} + (a - 1)\frac{\bar{k}_{\cdot}^2}{n^2}\right)\left(\frac{\bar{k}_{\cdot}^2 SSA_m}{n}\right)\right.$$

$$\left. - \left(\frac{(a - 1)\bar{k}_{\cdot}^2}{n}\right)\left(\frac{SSE}{\sigma_e^4} + \frac{\bar{k}_{\cdot}^2 SSA_m}{n^2}\right)\right]$$

$$= \frac{\sigma_e^4}{(N-a)(a-1)\bar{k}_.^2}\left[\frac{(N-a)\bar{k}_.^2 SSA_m}{n\sigma_e^4} - \frac{(a-1)\bar{k}_.^2 SSE}{n\sigma_e^4}\right]$$

$$= \frac{1}{n}\left[\frac{SSA_m}{a-1} - \frac{SSE}{N-a}\right]$$

$$= \frac{1}{n}\left[MSA_m - MSE\right] \quad . \qquad (A124)$$

Using (A117) through (A120) with (A68), (A69), and (A70) yields

$$v(\hat{\sigma}_{3e}^2) = \frac{2s_{3:11}}{\Delta_3} = \frac{2\sigma_e^4}{N-a} , \qquad (A125)$$

$$v(\hat{\sigma}_{3a}^2) = \frac{2s_{3:22}}{\Delta_3}$$

$$= \frac{2\sigma_e^4\left[\frac{N-a}{\sigma_e^4} + (a-1)\frac{\bar{k}_.^2}{n^2}\right]}{(N-a)(a-1)\bar{k}_.^2}$$

$$= \frac{2}{n^2}\left[\frac{(\sigma_e^2 + n\sigma_a^2)^2}{a-1} + \frac{\sigma_e^4}{N-a}\right] , \qquad (A126)$$

and $\operatorname{cov}(\hat{\sigma}_{3e}^2, \hat{\sigma}_{3a}^2) = \frac{-2s_{3:12}}{\Delta_3}$

$$= \frac{-2\sigma_e^4(a-1)\bar{k}_.^2}{n(N-a)(a-1)\bar{k}_.^2}$$

$$= \frac{-2\sigma_e^4}{n(N-a)} , \qquad (A127)$$

where (A125), (A126), and (A127) are the variances and covariance of the usual AOV estimators under balance.

4. MINQUE with $\mu \neq 0$

The results in (A123) and (A124) demonstrate that the MINQUE estimators, being a special case of the MIVQUE estimators, reduce to the usual AOV estimators under balance.

For $\ell_1 = \ell_2 = \cdots = \ell_a = \bar{\ell}_.$ in (A71) and $\sigma_e^2 = \sigma_a^2 = 1$ in (A117) through (A120) we get

$$s_{4:11} = (a - 1)\bar{\ell}_.^2 , \tag{A128}$$

$$s_{4:12} = \frac{(a - 1)\bar{\ell}_.^2}{n} = (a - 1)\bar{\ell}_.(1 - \bar{\ell}_.) , \tag{A129}$$

$$s_{4:22} = N - a + (a - 1)\frac{\bar{\ell}_.^2}{n^2}$$

$$= N - a + (a - 1)(1 - \bar{\ell}_.)^2 , \tag{A130}$$

and
$$\Delta_4 = (N - a)s_{4:11} , \tag{A131}$$

using the identity $\dfrac{\bar{\ell}_.}{n} = 1 - \bar{\ell}_.$.

Under balance (A89), (A95), and (A93) become

$$v(u_{4:1}) = 2\left[a\frac{\bar{\ell}_.^4}{\bar{k}_.^2} + 2\frac{\bar{\ell}_.^4}{\bar{k}_.^2} - 4\frac{\bar{\ell}_.^4}{\bar{k}_.^2} + \frac{\bar{\ell}_.^4}{\bar{k}_.^2} + 2\frac{\bar{\ell}_.^4}{\bar{k}_.^2} - 4\frac{\bar{\ell}_.^4}{\bar{k}_.^2} + 2\frac{\bar{\ell}_.^4}{\bar{k}_.^2}\right]$$

$$= 2(a - 1)\frac{\bar{\ell}_.^4}{\bar{k}_.^2} , \tag{A132}$$

$$\operatorname{cov}(u_{4:1},u_{4:2}) = 2\left[-(a - 1)\frac{\bar{\ell}_.^4}{\bar{k}_.^2} + a\frac{\bar{\ell}_.^3}{\bar{k}_.^2} - 3\frac{\bar{\ell}_.^3}{\bar{k}_.^2} + \frac{\bar{\ell}_.^3}{\bar{k}_.^2} - \frac{\bar{\ell}_.^3}{\bar{k}_.^2} + 2\frac{\bar{\ell}_.^3}{\bar{k}_.^2}\right]$$

$$= 2(a - 1)\left(\frac{\bar{\ell}^3_.}{\bar{k}^2_.} - \frac{\bar{\ell}^4_.}{\bar{k}^2_.}\right)$$

$$= 2(a - 1)\ \frac{\bar{\ell}^3_.}{\bar{k}^2_.}\ (1 - \bar{\ell}_.)\ , \tag{A133}$$

and
$$v(u_{4:2}) = 2\left[-(a - 1)\frac{\bar{\ell}^4_.}{\bar{k}^2_.} - 2(a - 1)\left(\frac{\bar{\ell}^3_.}{\bar{k}^2_.} - \frac{\bar{\ell}^4_.}{\bar{k}^2_.}\right) + N\sigma^4_e\right.$$

$$\left. + 2N\sigma^2_e\sigma^2_a + nN\sigma^4_a - a\frac{n^2}{\bar{k}^2_.} + a\frac{\bar{\ell}^2_.}{\bar{k}^2_.} - 2\frac{\bar{\ell}^2_.}{\bar{k}^2_.} + \frac{\bar{\ell}^2_.}{\bar{k}^2_.}\right]$$

$$= 2\left[(a - 1)\left(\frac{\bar{\ell}^2_.}{\bar{k}^2_.} - 2\frac{\bar{\ell}^3_.}{\bar{k}^2_.} + \frac{\bar{\ell}^4_.}{\bar{k}^2_.}\right) + N\sigma^4_e + 2N\sigma^2_e\sigma^2_a + nN\sigma^4_a\right.$$

$$\left. - a(\sigma^2_e + n\sigma^2_a)^2\right]$$

$$= 2\left[(a - 1)\ \frac{\bar{\ell}^2_.}{\bar{k}^2_.}\ (1 - \bar{\ell}_.)^2 + (N - a)\sigma^4_e\right]\ , \tag{A134}$$

remembering that $\bar{k}_. = \dfrac{n}{\sigma^2_e + n\sigma^2_a}$.

Then substituting (A128) through (A134) into (A79), (A80) and (A81) we get

$$v(\hat{\sigma}^2_{4e}) = \frac{2}{\Delta^2_4}\left[\left\{(a - 1)\bar{\ell}_.(1 - \bar{\ell}_.)\right\}^2\left\{(a - 1)\frac{\bar{\ell}^4_.}{\bar{k}^2_.}\right\}\right.$$

$$+ \left\{(a - 1)\bar{\ell}^2_.\right\}^2\left\{(a - 1)\frac{\bar{\ell}^2_.}{\bar{k}^2_.}(1 - \bar{\ell}_.)^2 + (N - a)\sigma^4_e\right\}$$

$$\left. - 2\left\{(a - 1)\bar{\ell}^2_.\right\}\left\{(a - 1)\bar{\ell}_.(1 - \bar{\ell}_.)\right\}\left\{(a - 1)\frac{\bar{\ell}^3_.}{\bar{k}^2_.}(1 - \bar{\ell}_.)\right\}\right]$$

$$= \frac{2(a-1)^2(N-a)\bar{\ell}^4_{\cdot}\sigma^4_e}{(N-a)^2(a-1)^2\bar{\ell}^4_{\cdot}} = \frac{2\sigma^4_e}{N-a} \quad , \tag{A135}$$

$$v(\hat{\sigma}^2_{4a}) = \frac{2}{\Delta^2_4}\Big[\Big\{N-a+(a-1)(1-\bar{\ell}_{\cdot})^2\Big\}^2\Big\{(a-1)\frac{\bar{\ell}^4_{\cdot}}{\bar{k}^2_{\cdot}}\Big\}$$

$$+\Big\{(a-1)\bar{\ell}_{\cdot}(1-\bar{\ell}_{\cdot})\Big\}^2\Big\{(a-1)\frac{\bar{\ell}^2_{\cdot}}{\bar{k}^2_{\cdot}}(1-\bar{\ell}_{\cdot})^2+(N-a)\sigma^4_e\Big\}$$

$$-2\Big\{(a-1)\bar{\ell}_{\cdot}(1-\bar{\ell}_{\cdot})\Big\}\Big\{N-a+(a-1)(1-\bar{\ell}_{\cdot})^2\Big\}$$

$$\times\Big\{(a-1)\frac{\bar{\ell}^3_{\cdot}}{\bar{k}^2_{\cdot}}(1-\bar{\ell}_{\cdot})\Big\}\Big]$$

$$= \frac{2\Big[(N-a)^2(a-1)\frac{\bar{\ell}^4_{\cdot}}{\bar{k}^2_{\cdot}}+(N-a)(a-1)^2\bar{\ell}^2_{\cdot}(1-\bar{\ell}_{\cdot})^2\sigma^4_e\Big]}{(N-a)^2(a-1)^2\bar{\ell}^4_{\cdot}}$$

$$= \frac{2}{(a-1)k^2_{\cdot}} + \frac{2\sigma^4_e}{n^2(N-a)}$$

$$= \frac{2}{n^2}\Big[\frac{(\sigma^2_e+n\sigma^2_a)^2}{a-1}+\frac{\sigma^4_e}{N-a}\Big] , \tag{A136}$$

and $$\mathrm{cov}(\hat{\sigma}^2_{4e},\hat{\sigma}^2_{4a}) = \frac{2}{\Delta^2_4}\Big[-\Big\{(a-1)\bar{\ell}_{\cdot}(1-\bar{\ell}_{\cdot})\Big\}\Big\{N-a+(a-1)(1-\bar{\ell}_{\cdot})^2\Big\}\Big\{(a-1)\frac{\bar{\ell}^4_{\cdot}}{\bar{k}^2_{\cdot}}\Big\}$$

$$-\Big\{(a-1)\bar{\ell}^2_{\cdot}\Big\}\Big\{(a-1)\bar{\ell}_{\cdot}(1-\bar{\ell}_{\cdot})\Big\}\Big\{(a-1)\frac{\bar{\ell}^2_{\cdot}}{\bar{k}^2_{\cdot}}(1-\bar{\ell}_{\cdot})^2+(N-a)\sigma^4_e\Big\}$$

$$+\Big(\Big\{(a-1)\bar{\ell}^2_{\cdot}\Big\}\Big\{N-a+(a-1)(1-\bar{\ell}_{\cdot})^2\Big\}+\Big\{(a-1)\bar{\ell}_{\cdot}(1-\bar{\ell}_{\cdot})\Big\}^2\Big)$$

$$\times\Big\{(a-1)\frac{\bar{\ell}^3_{\cdot}}{\bar{k}^2_{\cdot}}(1-\bar{\ell}_{\cdot})\Big\}\Big]$$

$$= \frac{2\left[-(N - a)(a - 1)^2\bar{\ell}_.^3(1 - \bar{\ell}_.)\sigma_e^4\right]}{(N - a)^2(a - 1)^2\bar{\ell}_.^4}$$

$$= \frac{-2\sigma_e^4}{n(N - a)} \quad . \tag{A137}$$

Equations (A135), (A136), and (A137) equal respectively (A125), (A126), and (A127) and are the variances and covariance under balance of the usual AOV estimators.